# Advanced Algorithm

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Lecture 10: LP-based Approximation Algorithm (continue)

# Weighted Set Cover problem - LP rounding

- Ref: Approximation Algorithm - Chapter 14

$$\begin{array}{ll} min & \sum_{S\in\mathfrak{S}} c(S)x_S \\ s.t & \sum_{S:e\in S} x_S \geq 1 \quad \forall e \in U \\ & x_S \in \{0,1\} \end{array}$$

- LP-relaxation

$$\begin{array}{ll} min & \sum_{S\in\mathfrak{S}} c(S)x_S \\ s.t & \sum_{S:e\in S} x_S \geq 1 \quad \forall e \in U \\ & x_S \geq 0 \end{array}$$

- LP rounding analysis
- randomized algorithm

# LP duality

**Theorem (Weak duality theorem)**

*If $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ are feasible solutions for the primal and dual program, respectively, then, $\sum_j c_j x_j \geq \sum_i b_i y_i$.*

**Theorem (Complementary slackness conditions)**

*Let $x$ and $y$ be primal and dual feasible solutions, respectively. Then, $x$ and $y$ are both optimal iff all of the following conditions are satisfied:*
*Primal complementary slackness conditions*
*For each $1 \leq j \leq n$: either $x_j = 0$ or $\sum_i a_{ij} y_i = c_j$;*
*Dual complementary slackness conditions*
*For each $1 \leq i \leq m$: either $y_i = 0$ or $\sum_j a_{ij} x_j = b_i$.*

# Weighted Set Cover problem - $f$ frequency

- Ref: Approximation Algorithm - Chapter 15
- Primal-dual Schema
- Let $x$ be a feasible solution of prime integer program, $y$ be a feasible solution of dual LP. If $x, y$ satisfy the following conditions
    - $\forall j$ : either $x_j = 0$ or $c_j/\alpha \le \sum_i a_{ij} y_i \le c_j$;
    - $\forall i$ : either $y_i = 0$ or $b_i \le \sum_j a_{ij} x_j \le \beta \cdot b_i$;
- then $\sum_j c_j x_j \le \alpha \cdot \beta \cdot \sum_i b_i y_i$.
- $f$-approximation
- integrality gap

# LP-based approximation

- State the problem as an integer programming
- Linear relaxation: integer program $\rightarrow$ linear program
  - Maximize problem: OPT $\uparrow$; Minimize problem: OPT $\downarrow$
- Method 1: linear-rounding
- Method 2: primal-dual schema
- Analysis: integrality gap
- For minimize problem, $ALGO \leq \alpha OPT_{LP} \leq \alpha OPT_{IP}$

# Homework

- Approximation Algorithm - Problem 13.2, page 116
- (optional) Approximation Algorithm - Problem 15.5, page 129 (matching problem)

# Lecture 10-2: Semidefinite Programming

# Semidefinite matrix

**Definition**

Let $A$ be a real, symmetric $n \times n$ matrix. A is positive semidefinite if $\forall x \in R^n, x^T Ax \geq 0$.

- $\forall x \in R^n, x^T Ax \geq 0$.
- All eigenvalues of $A$ are nonnegative.
- There is an $n \times n$ real matrix $W$, such that $A = W^T W$.
- $\exists$ real vectors $v_1, \cdots, v_n$, such that $A_{ij} =< v_i, v_j >$.

# Semidefinite Programming problem (SDP)

**Definition**

Semidefinite Programming problem Let $C, D_1, \cdots, D_k \in M_n$ be $n \times n$ real symmetric matrices. The following is a statement of the general semidefinite programming problem.

$$\begin{array}{ll} max & tr(C^T Y) \\ s.t & tr(D_i^T Y) \leq d_i \\ & Y \succeq 0 \\ & Y \in M_n \end{array}$$

- $tr(A^T B) = A \bullet B = \sum_{i,j} a_{ij} b_{ij}$

Algorithm for SDP:

- For any $\epsilon > 0$, SDP can be solved within an additive error of $\epsilon$ in time poly$(n, 1/\epsilon)$.
- Based on ellipsoid algorithm

# Vector program

## Definition

Vector program Let $C, D^1, \cdots, D^k \in M_n$ be $n \times n$ real symmetric matrices. The following is a statement of the general vector programming problem.

$$\begin{array}{ll} max & \sum_{i,j} C_{ij}(v_i \cdot v_j) \\ s.t & \sum_{i,j} D^{\ell}_{ij}(v_i \cdot v_j) \leq d_\ell \ (\forall \ell) \\ & v_i \in \mathbb{R}^n \ (\forall i) \end{array}$$

# MAX-CUT Problem

- Ref: Approximation Algorithm - Chapter 26
- MAX-CUT problem: given an undirected graph $G = (V, E)$ with non-negative edge weights, find a partition $(S, \bar{S})$ of $V$ so as to maximize the total weight of edges in this cut, that is, edges that have one endpoint in $S$ and one endpoint in $\bar{S}$.

$$\begin{array}{ll} max & \frac{1}{2}\sum_{i,j} w_{ij}(1 - y_i y_j) \\ s.t & y_i \in \{-1, 1\} \end{array}$$

# MAX-CUT Problem

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$$\begin{array}{ll} max & \frac{1}{2}\sum_{i,j} w_{ij}(1 - y_i y_j) \\ s.t & y_i^2 = 1 \\ & y_i \in \mathbb{Z} \end{array}$$

- SDP relaxation

$$\begin{array}{ll} max & \frac{1}{2}\sum_{i,j} w_{ij}(1 - v_i \cdot v_j) \\ s.t & v_i \cdot v_i = 1 \\ & v_i \in \mathbb{R}^n \end{array}$$

- 0.87856-approximation ratio

Lecture 10-3: Hardness of Approximation

# Gap-introducing reduction

- Ref: Approximation Algorithm, Chapter 29
- Gap-introducing reduction
- Let $A$ be a maximization optimization problem, consider a reduction from a NPC problem $B$ to $A$
- $\forall$ B's instance $I_B \rightarrow$ A's instance $I_A$ such that
  - If $I_B$ is satisfiable, then $OPT(I_A) \geq m$;
  - If $I_B$ is not satisfiable, then $OPT(I_A) \leq \alpha \cdot m$.
- Example: TSP, k-center problem, etc

# PCP

- Ref: Approximation Algorithm, Chapter 29
- PCP: probabilistically checkable proof systems (1992)
- PCP($r$: number of random bits, $q$ number of query bits)
- Verifier: polynomial time Turing machine, can query at most $q$ bits in the proof

## Definition (PCP)

A language $L \in \text{PCP}(r, q)$ if there exists a verifier $V$, such that on any input $x$, $V$ with a random string of $O(r)$ bits, can query at most $O(q)$ bits in a proof and

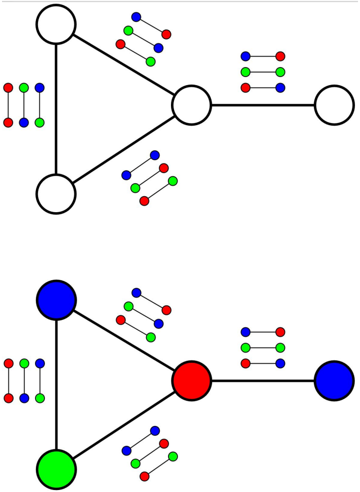
- if $x \in L$, then $\exists$ proof $y$, $V$ accepts with probability 1;
- if $x \notin L$, then $\forall$ proof $y$, $V$ accepts with probability $< 1/2$.

# PCP theorem

- some trivial relation
  - PCP(poly(n),0) = co-RP
  - PCP($\log n$,0) = P
  - PCP(0, $poly(n)$) = NP
- PCP Theorem: NP = PCP($\log n$, 1) by Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, Mario Szegedy (1998)
- Hardness results based on PCP (page 309)
- Hardness of MAX-3SAT

# Unique Game Conjecture

- Proposed by Subhash Khot in 2002



- $(c, s)$*-gap label-cover problem with unique constraints* is the following promise problem $(L_{yes}, L_{no})$:
  $L_{yes} = \{G$: Some assignment satisfies at least a c-fraction of constraints in $G\}$;
  $L_{no} = \{G$: Every assignment satisfies at most an s-fraction of constraints in $G\}$.

  where $G$ is an instance of the label cover problem with unique constraints.
- Unique Games Conjecture: for every sufficiently small pair of constants $\varepsilon, \delta > 0$, there exists a constant $k$ such that the $(1-\delta, \varepsilon)$-gap label-cover problem with unique constraints over alphabet of size $k$ is NP-hard.
- Inapproximation result based on UGC versus P$\neq$NP

| Problem | Poly-time approx. | NP hardness | UG hardness |
|---|---|---|---|
| Max 2-SAT | 0.940 | $0.954+\varepsilon$ | $0.940+\varepsilon$ |
| Max Cut | 0.878 | $0.941+\varepsilon$ | $0.878+\varepsilon$ |
| Vertex Cover | 2 | $1.360-\varepsilon$ | $2-\varepsilon$ |